A simplified Tikhonov regularization method for determining the heat source

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ARTICLE INFO

Received 29 November 2008
Received in revised form 4 February 2010
Accepted 12 February 2010
Available online 2 March 2010

Keywords:
Heat source
Simplified Tikhonov
Error estimate

ABSTRACT

This paper is to discuss the inverse problem of determining a spacewise dependent heat source in one-dimensional heat equation in a bounded domain where data is given at some fixed time. This problem is ill-posed, i.e., the solution (if it exists) does not depend continuously on the data. The regularization solution is given by a simplified Tikhonov regularization. For this regularization solution, the Hölder type stability estimate between the regularization solution and the exact solution is obtained. Numerical examples show that the regularization method is effective and stable.

1. Introduction

With development of society and economics, groundwater pollution has become a serious threat to the environment. The government has to take some measures to prevent the groundwater from further contaminations. But the cost of cleanup for polluted aquifers is staggering, and in many cases it is hard to identify which companies are responsible for the contamination, due to lack of tools to discover the pollution sources. So it is necessary to try to give more concrete information of the characteristics (location, magnitude, and duration of activity) of specific groundwater pollution sources. As we know, most attempts at quantifying contaminant transport rely on mathematical methods. Since the data can not be measured by direct ways in many cases, we are always encountering inverse problems of deciding unknown source. It is well known that under suitable hypotheses, solute transportation in a homogeneous groundwater flow can be characterized via the following one-dimensional (1D) heat equation [see \cite{1,2}]:

\begin{equation}
\frac{u_t}{u} + v u_x - a_L u_{xx} + \lambda u = \frac{f}{\eta_e}, \quad 0 < x < l, \quad 0 < t < T.
\end{equation}

where \( l > 0 \) denotes a bound for a studied region, \( T > 0 \) represents a final moment; \( u = u(x, t) \): concentration at time \( t \) and point \( x \), dimension: \( ML^{-3} \); \( v \): actual flow velocity of groundwater, dimension: \( LT^{-1} \); \( a_L \): longitudinal dispersivity, dimension: \( L \); \( \lambda \): attenuation coefficient, dimension: \( T^{-1} \); \( \eta_e \): effective porosity, no dimension; and \( f = f(x) \): average concentration of pollutants seeping into the aquifer per unit time, which represents the magnitude of pollution sources, dimension: \( ML^{-3}T^{-1} \).

In this paper, we consider standard equation of \((1.1)\) as follows:

\* The project is supported by the National Natural Science Foundation of China (No.10671085).
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doi:10.1016/j.apm.2010.02.020
If the function $f(x)$ is known, we can work out the concentration distribution $u = u(x, t)$ by the initial boundary value. However, the problem here is that the source magnitude function $f(x)$ is unknown, which needs to be decided by some additional data. The additional data discussed in this study are observations at a final moment $t = 1$ given as follows:

$$u(x, 1) = g(x), \quad 0 \leq x \leq 1. \quad (1.3)$$

This problem is called the identification of heat source, that has received considerable attention from many researchers in a variety of fields using different methods since 1970. If the heat source has the form of $f = f(u)$, the inverse source problem was studied by [3]. In [4], the authors considered the heat source as a function of both space and time, but is additive or separable. But many researchers viewed the source as a function of space or time only. In [5,6], the authors determined the heat source dependent on one variable in a bounded domain by the boundary-element method and an iterative algorithm. In [7], the authors identified the heat source to be time-dependent only by the method of fundamental solution.

As we know, there are lots of researches on identification of heat source adopted numerical algorithms, such as the mollification method [8,9], the finite difference method [10,11], the meshless method [12] and the conditional stability [13,14]. But by regularization method, there are few papers with strict theoretical analysis on identifying the heat source. Recently, in [15,16], the authors identified the heat source by the Fourier regularization method, and obtained the error estimates.

We want to determine the heat source $f(x)$ from the data $g(x)$ by a simplified Tikhonov regularization method. This method was first introduced by Carasso in [17], then the idea of this method has been successfully used for solving various types of ill-posed problems. In [18], the author considered the inverse heat conduction problem. In [19,20], the author solved a spherically symmetric inverse heat conduction problem.

This paper is organized as follows: In Section 2, we simply analyze the ill-posedness of problem (1.2)–(1.3) and give some auxiliary results. In Section 3, we give the regularization solution by using the simplified Tikhonov regularization and give the stable error estimate between the regularization solution and the exact solution. To present a clear overview of the method, in Section 4, we give several numerical examples including both smooth and non-smooth cases. Section 5 puts an end to this paper with a brief conclusion.

2. Some auxiliary results

By separating variables, we obtain the solution of problem (1.2)–(1.3) as follows:

$$u(x, t) = \sum_{n=1}^{\infty} \frac{1 - e^{-n^2 \pi^2 t}}{n^2 \pi^2} \langle f, X_n \rangle X_n, \quad (2.1)$$

where

$$\left\{ X_n = \sqrt{2} \cos n \pi x, (n = 1, 2 \ldots) \right\} \quad (2.2)$$

is an orthogonal basis in $L^2(0, 1)$, and

$$\langle f, X_n \rangle = \sqrt{2} \int_0^1 f(x) \cos(n \pi x) dx. \quad (2.3)$$

By the supplementary condition, we define the operator $K : f \to g$, then we have

$$g(x) = Kf(x) = \sum_{n=1}^{\infty} \langle g, X_n \rangle X_n = \sum_{n=1}^{\infty} \frac{1 - e^{-n^2 \pi^2}}{n^2 \pi^2} \langle f, X_n \rangle X_n. \quad (2.4)$$

It is easy to see that $K$ is a linear compact operator (see [21]), and the singular values $\{ \sigma_n \}_{n=1}^{\infty}$ of $K$ satisfy

$$\sigma_n = \frac{1 - e^{-n^2 \pi^2}}{n^2 \pi^2}, \quad (2.5)$$

and

$$\langle g, X_n \rangle = \frac{1 - e^{-n^2 \pi^2}}{n^2 \pi^2} \langle f, X_n \rangle (X_n, X_n), \quad (2.6)$$

i.e.,

$$\langle f, X_n \rangle = \sigma_n^{-1} \langle g, X_n \rangle. \quad (2.7)$$

Therefore

$$f(x) = K^{-1} g(x) = \sum_{n=1}^{\infty} \frac{1}{\sigma_n} \langle g, X_n \rangle X_n = \sum_{n=1}^{\infty} \frac{n^2 \pi^2}{1 - e^{-n^2 \pi^2}} \langle g, X_n \rangle X_n. \quad (2.8)$$
Note that \( \frac{1}{n^2} = o(n^2) \) as \( n \to \infty \), thus the exact data function \( g(x) \) must satisfy the property that \( (g, X_\alpha) \) decays rapidly as \( o(n^{-2}) \). But in applications, the input data \( g(x) \) can only be measured and never be exact. We assume the measured data function \( g_\delta(x) \in L^2(0, 1) \) and satisfies
\[
\|g - g_\delta\|_{L^2(0, 1)} \leq \delta, \tag{2.9}
\]
where the constant \( \delta > 0 \) represents a noise level. So we can not expect it has the same decay rate in \( L^2(0, 1) \). Thus it is impossible to gain the heat source using classical methods. In the following section, we will use a simplified Tikhonov regularization method to deal with the ill-posed problem. Before doing that, we impose an a priori bound on the heat source, i.e.,
\[
\|f\|_{H^p(0, 1)} \leq E, \quad p \geq 0. \tag{2.10}
\]
where \( E > 0 \) is a constant, \( \| \cdot \|_{H^p(0, 1)} \) denotes the norm in Sobolev space, \( H^p(0, 1) \) is defined by [22] as following:
\[
\|f\|_{H^p(0, 1)} = \left( \sum_{n=1}^{\infty} (1 + n^2)^p |(f, X_n)|^2 \right)^{\frac{1}{2}}. \tag{2.11}
\]
Now we give some important lemmas which are very useful for our main conclusion.

**Lemma 2.1.** For \( n \geq 1 \), there holds
\[
\frac{1}{1 - e^{-n^2\pi^2}} \leq 2. \tag{2.12}
\]

**Proof.** The proof is easy and is omitted. \( \square \)

**Lemma 2.2.** For \( 0 < \alpha < 1 \), there holds the following inequalities:
\[
\sup_{n \geq 1} \left( \frac{1}{1 + \alpha^2 n^2} (1 + n^2)^{\frac{1}{2}} \right) \leq \max(\alpha^2, \alpha^2), \tag{2.13}
\]
\[
\sup_{n \geq 1} \left( \frac{n^2 \pi^2}{(1 - e^{-n^2\pi^2})(1 + \alpha^2 n^2)} \right) \leq \frac{2\pi^2}{\alpha^2}. \tag{2.14}
\]

**Proof.** Let
\[
G(n) := \left( \frac{1}{1 + \alpha^2 n^2} (1 + n^2)^{\frac{1}{2}} \right).
\]
The proof of (2.13) can be separated from two cases:

**Case 1.** For large values of \( n \), i.e., \( n \geq n_0 := \frac{1}{2} \), we get
\[
G(n) \leq (1 + n^2)^{\frac{1}{2}} \leq n^{-p} \leq n_0^{-p} = \alpha^p. \tag{2.16}
\]

**Case 2.** \( 1 \leq n < n_0 \), we obtain
\[
G(n) = \frac{\alpha^2 n^2}{1 + \alpha^2 n^2} (1 + n^2)^{\frac{1}{2}} \leq \alpha^2 n^2 (1 + n^2)^{\frac{1}{2}} \leq \alpha^2 n^{2-p}. \tag{2.17}
\]
If \( 0 < p \leq 2 \), above inequality becomes into
\[
G(n) \leq \alpha^2 n^{2-p} < \alpha^2 n_0^{2-p} = \alpha^2 \alpha^{p-2} = \alpha^p. \tag{2.18}
\]
Else if \( p > 2 \), we get
\[
G(n) \leq \alpha^2 n^{p-2} \leq \alpha^2. \tag{2.19}
\]
Combining (2.16) with (2.18) and (2.19), the first inequality equation holds. Let
\[
B(n) := \frac{n^2 \pi^2}{(1 - e^{-n^2\pi^2})(1 + \alpha^2 n^2)} , \quad D(n) := \frac{n^2 \pi^2}{1 - e^{-n^2\pi^2}}. \tag{2.20}
\]
The proof of (2.14) can also be divided into two cases:

**Case 1.** \( 1 \leq n \leq n_0 := \frac{1}{2} \), we have
\[
D(n) \leq D \left( \frac{1}{\alpha} \right) \leq \frac{2\pi^2}{\alpha^2} , \quad \text{if } 0 < \alpha < 1. \tag{2.21}
\]
So
$$B(n) \leq D(n) \leq \frac{2\pi^2}{\alpha^2}.$$  \hspace{0.5cm} (2.22)

**Case 2.** If $n > n_0$, we can get
$$D(n) \leq 2\pi^2 n^2,$$  \hspace{0.5cm} (2.23)
and
$$B(n) \leq \frac{2\pi^2 n^2}{1 + \alpha^2 n^4}.$$  \hspace{0.5cm} (2.24)

Let
$$L(n) := \frac{2\pi^2 n^2}{1 + \alpha^2 n^4},$$  \hspace{0.5cm} (2.25)
then
$$L'(n) = \frac{4\pi^2 n(1 - \alpha^2 n^4)}{(1 + \alpha^2 n^4)^2}.$$  \hspace{0.5cm} (2.26)

Setting $L'(n) = 0$, we can obtain $n_1 = \frac{1}{\sqrt{\alpha}}$. It is easy to see that $n_1 = \frac{1}{\sqrt{\alpha}}$ is a maximal value point of $L(n)$. So
$$L(n) \leq \frac{2\pi^2 n_1^2}{1 + \alpha^2 n_1^4} \leq 2\pi^2 n_1^2 = \frac{2\pi^2}{\alpha} \leq \frac{2\pi^2}{\alpha^2}.$$  \hspace{0.5cm} (2.27)

Combining (2.22) with (2.27), inequality (2.14) holds. □

**3. A simplified Tikhonov regularization method**

From (2.4), we know problem (1.2)–(1.3) can be formulated as an operator equation:
$$(Kf)(x) = g(x).$$  \hspace{0.5cm} (3.1)

Since problem (3.1) is an ill-posed problem, we give an approximate solution of $f(x)$ by a Tikhonov regularization method which minimizes the quantity
$$\|Kf - g\|^2 + \alpha^2\|f\|^2.$$  \hspace{0.5cm} (3.2)

Then by Theorem 2.12 in [22], the unique solution of the minimization problem (3.2) is equal to solve the following normal equation:
$$K^*Kf_\alpha(x) + \alpha^2 f_\alpha(x) = K^*g_\alpha(x),$$  \hspace{0.5cm} (3.3)

i.e.,
$$f_\alpha(x) = [K^*K + \alpha^2 I]^{-1}K^*g_\alpha(x).$$  \hspace{0.5cm} (3.4)

Because $K$ is a linear self-adjoint compact operator, i.e., $K^* = K$, we have the equivalent form
$$f_\alpha(x) = [K^2 + \alpha^2 I]^{-1}Kg_\alpha(x).$$  \hspace{0.5cm} (3.5)

We define function of a compact self-adjoint operator $K$ by the spectral mapping theorem in the following way:

**Definition 3.1** [23]. If $f(x)$ is a real-valued continuous function on the spectrum $\sigma(K)$, we define $f(K)$ by
$$f(K)x = \sum_n f(\lambda_n)(x, \omega_n)\omega_n,$$  \hspace{0.5cm} (3.6)

where $K$ is a compact self-adjoint, $\lambda_n \in \sigma(K)$, and $\omega_n$ are the corresponding orthogonal eigenvectors.

So we obtain
$$f_\alpha(x) = \sum_{n=1}^\infty \frac{1 - e^{-\alpha^2 n^2}}{\alpha^2 + (1 - e^{-\alpha^2 n^2})^2} (g_{\lambda_n}, X_n)X_n$$  \hspace{0.5cm} (3.7)

$$= \sum_{n=1}^\infty \frac{1 - e^{-\alpha^2 n^2}}{1 + \alpha^2 (1 - e^{-\alpha^2 n^2})^2} (g_{\lambda_n}, X_n)X_n$$  \hspace{0.5cm} (3.8)

$$= \sum_{n=1}^\infty \frac{1 - e^{-\alpha^2 n^2}}{1 + \alpha^2 (1 - e^{-\alpha^2 n^2})^2} g_{\lambda_n}X_n.$$  \hspace{0.5cm} (3.9)
Comparing formula (2.8) with formula (3.9), we can find that the procedure consists in replacing the unknown $g(x)$ with an appropriately filtered noised data $g_{i,n}(x)$. The filter in (3.9) attenuates the coefficient $g_{i,n}(x)$ in a manner consistent with the goal of minimizing quantity (3.2). By this idea, we can use a much better filter $\frac{1}{1 + (\cdot)^2}$ and give another approximation $f_{\alpha,n}(x)$ of the solution $f(x)$.

We define a regularization approximate solution of problem (1.2)–(1.3) or (3.1) for noisy data $g_{i,n}(x)$ which is called the simplified Tikhonov regularized solution of problem (1.2)–(1.3) or (3.1) as follows:

$$f_{\alpha,n}(x) := \sum_{n=1}^{\infty} \frac{n^2 \pi^2}{(1 - e^{-n^2 \pi^2})(1 + 2^n n^4)} (g_{i,n}, X_n) X_n.$$  \hfill (3.10)

**Theorem 3.2.** Let $f_{\alpha,n}(x)$ be the simplified Tikhonov approximation of the solution $f(x)$ of problem (1.2)–(1.3). Let $g_{i,n}(x)$ be measured data at $t = 1$ satisfying (2.9) and prior condition (2.10) holds for $p > 0$. If we select

$$\alpha = \left(\frac{\delta}{E}\right)^{\frac{1}{p^2}},$$ \hfill (3.11)

then the following estimate holds:

$$\|f(x) - f_{\alpha,n}(x)\|_{l^2(0,1)} \leq 2\pi^2 \delta^{\frac{1}{2p^2}} E^{\frac{1}{p^2}} \left(1 + \frac{1}{2\pi^2} \max \left(1, \left(\frac{\delta}{E}\right)^{\frac{2}{p^2}}\right)\right).$$ \hfill (3.12)

**Proof.** Due to the triangle inequality, we have

$$\|f(x) - f_{\alpha,n}(x)\|_{l^2(0,1)} = \sum_{n=1}^{\infty} \frac{n^2 \pi^2}{(1 - e^{-n^2 \pi^2})(1 + 2^n n^4)} (g_{i,n}, X_n) X_n \leq \sum_{n=1}^{\infty} \frac{n^2 \pi^2}{(1 - e^{-n^2 \pi^2})(1 + 2^n n^4)} (g_{i,n}, X_n) X_n + \sum_{n=1}^{\infty} \frac{n^2 \pi^2}{(1 - e^{-n^2 \pi^2})(1 + 2^n n^4)} (g_{i,n}, X_n) X_n = \sum_{n=1}^{\infty} \frac{n^2 \pi^2}{(1 - e^{-n^2 \pi^2})(1 + 2^n n^4)} (g_{i,n}, X_n) X_n.$$

Then we can find that

$$\|f(x) - f_{\alpha,n}(x)\|_{l^2(0,1)} \leq \sum_{n=1}^{\infty} \frac{n^2 \pi^2}{(1 - e^{-n^2 \pi^2})(1 + 2^n n^4)} (g_{i,n}, X_n) X_n + \sum_{n=1}^{\infty} \frac{n^2 \pi^2}{(1 - e^{-n^2 \pi^2})(1 + 2^n n^4)} (g_{i,n}, X_n) X_n = \sum_{n=1}^{\infty} \frac{n^2 \pi^2}{(1 - e^{-n^2 \pi^2})(1 + 2^n n^4)} (g_{i,n}, X_n) X_n.$$

**Remark 3.3.** If $0 < P \leq 2$, $\|f(x) - f_{\alpha,n}(x)\|_{l^2(0,1)} \leq (2\pi^2 + 1) \delta^{\frac{1}{2p^2}} E^{\frac{1}{p^2}} \to 0$, as $\delta \to 0$, else if $p > 2$, $\|f(x) - f_{\alpha,n}(x)\|_{l^2(0,1)} \leq (2\pi^2 + 1) \delta^{\frac{1}{2p^2}} E^{\frac{1}{p^2}} \to 0$, as $\delta \to 0$. Hence $f_{\alpha,n}(x)$ can be viewed as the approximation of the exact solution $f(x)$.

**Remark 3.4.** In practice, $\|f\|_{l^p(0,1)}$ is usually not known, so we do not obtain an exact prior bound $E$. However, if we select $\alpha = C\delta^{\frac{1}{p^2}}$, where $C$ is a positive constant, we can also obtain

$$\|f(x) - f_{\alpha,n}(x)\|_{l^2(0,1)} \leq C\delta^{\frac{1}{p^2}},$$ \hfill (3.13)

where the constant $C$ depends on $p$, $\|f\|_{l^p(0,1)}$.  

4. Numerical implementation

In this section, we describe a numerical implementation of the simplified Tikhonov regularization method. From (3.1), we know that

\[
(Kf)(x) = \sum_{n=1}^{\infty} (f, X_n) \frac{1 - e^{-n^2 \pi^2}}{n^2 \pi^2} X_n
\]

\[
= \int_0^1 2 \sum_{n=1}^{\infty} \frac{1 - e^{-n^2 \pi^2}}{n^2 \pi^2} f(s) \cos(n \pi x) \cos(n \pi x) ds = g(x).
\]

We do an approximate truncation for the series by choosing the sum of the front \( N \) terms and use trapezoid’s rule to approach the integral. After considering an equidistant grid \( 0 = x_0 < \ldots < x_M = 1 \) \( (x_i = \frac{i}{M}, \; i = 0, \ldots, M) \), we get

\[
2 \sum_{i=0}^{M} \sum_{n=1}^{N} \frac{1 - e^{-n^2 \pi^2}}{n^2 \pi^2} f(x_i) \cos(n \pi x_i) \cos(n \pi x_j) h = g(x_j),
\]

where

\[
h = \frac{1}{M}.
\]

So we obtain the matrix form of (4.3)

\[
AV = Z,
\]

where

\[
A_{ij} = 2 \sum_{n=1}^{N} \frac{1 - e^{-n^2 \pi^2}}{n^2 \pi^2} \cos(n \pi x_i) \cos(n \pi x_j) h,
\]

\[
V_j = f(x_j),
\]

\[
Z_i = g(x_i), \; \; i, j = 0, \ldots, M.
\]

The discretized simplified Tikhonov regularization solution is

\[
V_{x, \delta} = \frac{1}{1 + 2 \xi^2} A Z_{\delta},
\]

which corresponds to the continuous simplified Tikhonov regularization solution (3.10). We can obtain the vector \( Z_\delta = (Z_{x, \delta}) \in \mathbb{R}^{M+1} \) by adding a random distributed perturbation to the data \( Z_i = g(x_i) \), i.e.,

\[
Z_\delta = Z + \delta \text{ randn(size} (z).)
\]

The function “randn (·)” generates arrays of random numbers whose elements are normally distributed with mean 0, variance \( \sigma^2 = 1 \), and standard deviation \( \sigma = 1 \). “randn (size (z))” returns an array of random entries that is the same size as \( Z \). The noisy level \( \delta \) is computed according to

\[
\|Z - Z_\delta\|_2 := \sqrt{\frac{1}{M+1} \sum_{i=0}^{M} (Z_i - Z_{x, \delta})^2} = \delta.
\]

5. Numerical experiments

In this section, some numerical results are reported in order to show how the regularization method works. We give some examples including the continuous and the discontinuous cases. In the numerical experiments, we compute \( V_{x, \delta} \) by (4.9), and the relative error \( e_{r}(Z) \) is given by

\[
e_{r}(Z) := \frac{\|Z_{x, \delta} (\cdot) - Z(\cdot)\|_2}{\|Z(\cdot)\|_2},
\]

where \( \| \cdot \|_2 \) is defined by (4.11).

Sometimes, it is difficult to find a pair function \( (u, f) \) which satisfy the problem (1.2). We give the following numerical implementation: let the first-order derivative \( \frac{df(x)}{dx} \) and the second-order derivative \( \frac{d^2f(x)}{dx^2} \) of function \( f(x) \) be given approximately as

\[
\frac{df(x)}{dx} \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}, \quad \frac{d^2f(x)}{dx^2} \approx \frac{f(x + \Delta x) - 2f(x) + f(x - \Delta x)}{(\Delta x)^2}.
\]
Considering an equidistant grid $0 = x_0 < \ldots < x_M = 1$ (where $x_i = ih$, $h = \frac{1}{M}$, $i = 0, \ldots, M$), $0 = t_0 < t_1 < \cdots < t_M = 1$ ($t_j = j\tau$, $\tau = \frac{1}{M}$, $j = 0, \ldots, M$), we discretize the first equation of (1.2) with respect to the variable $x$ and $t$ as follows,

$$DU^{j+1} = CU^j + B, \quad 0 \leq j \leq M - 1,$$

where

$$D = \begin{pmatrix}
2 & -\frac{1}{2} & 0 & \cdots & 0 \\
-\frac{1}{2} & 2 & -\frac{1}{2} & \cdots & 0 \\
& \ddots & \ddots & \ddots & \ddots \\
& -\frac{1}{2} & 2 & -\frac{1}{2} & 0 \\
& & -\frac{1}{2} & 2 & \frac{1}{2}
\end{pmatrix}, \quad C = \begin{pmatrix}
0 & \frac{1}{2} & 0 & \cdots & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & \cdots & 0 \\
& \ddots & \ddots & \ddots & \ddots \\
& \frac{1}{2} & 0 & \frac{1}{2} & 0 \\
& & \frac{1}{2} & 0 & \frac{1}{2}
\end{pmatrix},$$

$$B = \left( \frac{1}{2}(u_0 + u_0^{-1}) + \tau f(x_1), \tau f(x_2), \ldots, \tau f(x_{M-2}), \frac{1}{2}(u_M + u_M^{-1}) + \tau f(x_{M-1}) \right)^T,$$

$$u_0^j = u(x_0, t_j), \quad u_0^{-1} = u(x_0, t_j+1), \quad u_M^j = u(x_M, t_j), \quad u_M^{-1} = u(x_M, t_{j+1}),$$

$$U^j = \left( u(x_1, t_j), u(x_2, t_j), \ldots, u(x_{M-2}, t_j), u(x_{M-1}, t_j) \right)^T, \quad t_{j+1} = \frac{1}{2}(t_j + t_{j+1}).$$

### Table 1

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### Table 3

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<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$e_r(f)$</td>
<td>0.0040</td>
<td>0.0041</td>
<td>0.0041</td>
<td>0.0047</td>
<td>0.0045</td>
<td>0.0042</td>
<td>0.0050</td>
<td>0.0045</td>
</tr>
</tbody>
</table>

**Fig. 1.** The comparison between the exact solution and its computed approximation with $M = 400$, $N = 7$ and $p = 2$ for $\delta = 0.1$. 

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<table>
<thead>
<tr>
<th>$N$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
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</table>
The discretized second equation of (1.2) with respect to \( x \) is
\[ U(x, 0) = 0, \quad 0 \leq i \leq M. \]

The discretized boundary condition is
\[
\begin{align*}
-D_x U_{0}^{i+\frac{1}{2}} + \frac{\tau}{2} \delta_x U_{0}^{i+\frac{1}{2}} &= \frac{h}{2} f(x_0), \quad 0 \leq j \leq M - 1, \\
D_x U_{M}^{i+\frac{1}{2}} + \frac{\tau}{2} \delta_x U_{M}^{i+\frac{1}{2}} &= \frac{h}{2} f(x_M), \quad 0 \leq j \leq M - 1,
\end{align*}
\]
\( (5.7) \)

where
\[
\begin{align*}
D_x U_{0}^{i+\frac{1}{2}} &= \frac{U_{1}^{i+\frac{1}{2}} - U_{0}^{i+\frac{1}{2}}}{h}, \\
\delta_x U_{0}^{i+\frac{1}{2}} &= \frac{U_{0}^{i+\frac{1}{2}} - U_{0}^{i-\frac{1}{2}}}{\tau}, \\
D_x U_{M}^{i+\frac{1}{2}} &= \frac{U_{M}^{i+\frac{1}{2}} - U_{M-1}^{i+\frac{1}{2}}}{h},
\end{align*}
\]
\( (5.8) \)
\( (5.9) \)

\( \text{Fig. 2.} \) The comparison between the exact solution and its computed approximation with \( M = 400 \), \( N = 7 \) and \( p = 2 \) for \( \delta = 0.05 \).

\( \text{Fig. 3.} \) The comparison between the exact solution and its computed approximation with \( M = 400 \), \( N = 7 \) and \( p = 2 \) for \( \delta = 0.01 \).
So the problem (1.2) is discretized as follows
\[
\begin{align*}
DU_{j+1} &= CU_j + B, \\
-D_x U_{j+1}^0 + \frac{h}{2} \delta_iU_{j+1}^1 &= \frac{b}{2} f(x_0), & 0 \leq j \leq M - 1, \\
-D_x U_{M+1}^k + \frac{h}{2} \delta_iU_{M+1}^1 &= \frac{b}{2} f(x_M). & 0 \leq j \leq M - 1.
\end{align*}
\] (5.10)

So firstly, we can get the measurable data \(g(x)\) by solving the direct problem (5.10) when the exact solution \(f(x)\) is given; Secondly, \(g_d(x)\) is obtained by (4.10) and the regularization solution \(f_a(x)\) is obtained according to (4.9).

In the four examples present below, the first example such an \(f(x)\) is available analytically of problem (1.2). The other three examples do not possess an explicit analytical expression. \(f(x)\) of the second example is an infinitely differentiable smooth heat source. \(f(x)\) of the third example is a piecewise smooth heat source. \(f(x)\) of the fourth example is a discontinuous heat source.

**Example 1.** It is easy to see that the function \(u(x, t) = \cos \pi x (1 - e^{-\pi^2 t})\) and the function \(f(x) = \pi^2 \cos \pi x\) are the exact solution of the problem (1.2). Consequently, the data function is \(g(x) = \pi \cos \pi x(1 - e^{-\pi^2}).\)
Example 2. We consider the following direct problem:

\[
\begin{align*}
    u_t - u_{xx} &= f(x), & 0 < x < 1, & 0 < t \leq 1, \\
    u(x, 0) &= 0, & 0 \leq x \leq 1, \\
    u_x(0, t) = u_x(1, t) &= 0, & 0 \leq t \leq 1,
\end{align*}
\]

with the heat source function

\[ f(x) = \sin 4\pi x, \quad 0 \leq x \leq 1. \]

(5.11)

In Table 1, according to (3.13), we take the variance of the parameter \( \alpha \) to compute the relative error between the exact solution and the regularization solution for Example 1. From Table 1, we can at least find two useful information. Firstly, the parameter \( \alpha \) has a regularization effect. A better parameter choice is \( \alpha = 0.2\alpha_0 \). Secondly, we can see that the quality of the numerical solution is not very sensitive to variations of the parameter \( \alpha \). Thus, in practice, it is relatively easy to find an appropriate value for \( \alpha \).

Considering Example 1, for studying the effect of increasing \( N \) and \( M \), we give Tables 2 and 3. From these tables, we find that \( M \) and \( N \) have little influence on the results when they become larger, i.e., the degree of ill-posedness of numerical

Fig. 6. The comparison between the exact solution and its computed approximation with \( M = 400, N = 7 \) and \( p = 2 \) for \( \delta = 0.00001 \).

Fig. 7. The comparison between the exact solution and its computed approximation with \( M = 400, N = 7 \) and \( p = 2 \) for \( \delta = 0.001 \).
problems does not increase with the refinement of the mesh used. Thus, we take $N = 7$ and $M = 400$ in the following examination.

**Example 3.** Consider a piecewise smooth heat source:

$$f(x) = \begin{cases} 
0, & 0 \leq x \leq 0.3, \\
5(x - 0.3), & 0.3 < x \leq 0.5, \\
-5(x - 0.7), & 0.5 < x \leq 0.7, \\
0, & 0.7 < x \leq 1.
\end{cases} \quad (5.13)$$
Example 4. Consider the following discontinuous case

\[
    f(x) = \begin{cases} 
        -1, & 0 \leq x \leq 0.25, \\
        1, & 0.25 < x \leq 0.5, \\
        -1, & 0.5 < x \leq 0.75, \\
        1, & 0.75 < x \leq 1.
    \end{cases}
\]  

(5.14)

Figs. 1–3 show the comparisons between the exact solution and its computed approximation with different noise level for Example 1. It can be seen from Figs. 1–3 that the numerical results are quite satisfactory. Even with the noise level up to \( \delta = 0.1 \), the numerical solutions are still in good agreement with the exact solution.

In Examples 2–4, since the direct problem with the heat source \( f(x) \) does not have an analytical solution, the data \( g(x) \) is obtained by solving the direct problem (5.10). Figs. 4–6 show the comparisons between the exact solution and its computed approximation with different noise level for Example 2. It can be seen that as the amount of noise \( \delta \) decreases, the regular-
ized solutions approximate better the exact solution. Figs. 7–9 indicate the comparisons between the exact solution and its computed approximation with different noise level for Example 3. Figs. 10–12 indicate the comparisons between the exact solution and its computed approximation with different noise level for Example 4. From Figs. 7–12, it can be seen that the numerical solution is less than that of Examples 1 and 2. It is not difficult to see that the well-known Gibbs phenomenon and the recovered data near the non-smooth and discontinuities points are not accurate. Taking into consideration the ill-posedness of the problem, the results presented in Figs. 7–12 are reasonable.

6. Conclusions

In this paper, a converge error estimate between the exact solution and the regularized solution has been obtained. The numerical results show that the simplified Tikhonov regularization method is accurate and reliable for identification of heat source dependent only spacial variable. Compared with other methods for identification of heat source, the Simplified Tikhonov regularization method considered in the present paper has the following advantages: (i) the theory analysis and the numerical implementation are simpler than the Tikhonov regularization method. (ii) convergence estimate is obtained. (iii) the method can also be extended to the non-homogeneous boundary conditions using linearity. (iv) the method can be applied to determining the heat source as $f(x,t) = f(t)u$, [24]. (v) the method can be adopted to determine the source of the wave and poisson equations. However, (iv) and (v) will be left for future work.

References


